MA 410, Fall 2011

Assignment 5

This homework is due *Monday*, October 17.

There are total 37 points in this assignment. 33 points is considered 100%. If you go over 33 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations. Bare answers will not earn you much.

This assignment covers sections 3.1–3.5 in O'Neill.

(1) [2pt] (3.1.4) For

$$C = \begin{pmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix}$$

show that C is orthogonal. Given points $p = (3, 1, -6)$, $q = (1, 0, 3)$, compute $C(p)$ and $C(q)$, and check that $C(p) \bullet C(q) = p \bullet q$.

(2) (3.1.1,2,3)

(a) [4pt] Let C be an orthogonal transformation, and T_a be a translation by vector a. Prove that $CT_a = T_{C(a)}C$.

(b) [3pt] Given isometries $F = T_aA$, $G = T_bB$ (here A, B are orthogonal transformations, T_a, T_b are translations), find the translation and the orthogonal part of FG and GF. (*Hint:* Use the previous item.)

(c) [3pt] Show that an isometry $F = TC$ has an inverse mapping F^{-1}, which is also an isometry. Find translation and orthogonal parts of F^{-1}. (*Hint:* Use general formula $(FG)^{-1} = G^{-1}F^{-1}$ and the item (a).)

(3) (3.1.5) In each case below decide whether F is an isometry of $\mathbb{R}^3$. If so, find its translation and orthogonal parts.

(a) [2pt] $F(P) = -p$,

(b) [2pt] $F(P) = (p \bullet a)a$, where $\|a\| = 1$,

(c) [3pt] $F(P) = (p_3 - 1, p_2 - 2, p_1 - 3)$,

(d) [2pt] $F(P) = (p_1, p_2, 1)$.

(4) [3pt] (3.2.3) Given the frame

$$e_1 = (2, 2, 1)/3, \quad e_2 = (-2, 1, 2)/3, \quad e_3 = (1, -2, 2)/3$$

at $p = (0, 1, 0)$ and the frame

$$f_1 = (1, 0, 1)/\sqrt{2}, \quad f_2 = (0, 1, 0), \quad f_3 = (1, 0, -1)/\sqrt{2}$$

at $q = (3, -1, 1)$, find a and C such that the isometry T_aC carries the frame e to the frame f.

— *see next page* —

(5) [3pt] As shown in the previous homework, the unit speed curve

$$\beta(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right)$$

is a circle. Find an isometry that carries circle $(\cos s, \sin s, 0)$ onto β (possibly with a shift of parameter).

(6) [3pt] (3.3.1) Prove

$$\operatorname{sgn}(FG) = \operatorname{sgn} F \cdot \operatorname{sgn} G = \operatorname{sgn}(GF).$$

Deduce $\operatorname{sgn} F = \operatorname{sgn} F^{-1}$.

(7) [3pt] (3.3.2) If H_0 is a fixed orientation-reversing isometry of $\mathbb{R}^3$, show that every orientation-reversing isometry has a unique expression H_0F, where F is orientation-preserving. (*Hint:* Use the previous problem.)

(8) [4pt] (3.4.2) Let $Y = (t, 1 - t^2, 1 + t^2)$ be a vector field on the helix

$$\alpha(t) = (\cos t, \sin t, 2t),$$

and let C be the orthogonal matrix

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Compute $\overline{\alpha} = C(\alpha)$ and $\overline{Y} = C*(Y)$, and check that

$$C*(Y') = \overline{Y}', \quad C*(\alpha'') = \overline{\alpha}'', \quad Y' \bullet \alpha'' = \overline{Y}' \bullet \overline{\alpha}''.$$